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L_p-Saturation of Some Modified Bernstein Operators

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1. INTRODUCTION

We determine the saturation class for the L_p-norms, $1 \leqslant p < \infty$, of the modified Bernstein operators M_n. These operators were first defined and studied by Durrmeyer [4] and later by Derriennic [1, 2, 3]. For a function $f \in L_1(I)$, with $I := [0, 1]$, $M_n f$ is given by

$$(M_n f)(x) := (n+1) \sum_{k=0}^{n} p_{nk}(x) \int_I p_{nk}(t) f(t)\, dt, \qquad n \in \mathbb{N},$$

where

$$p_{nk}(x) := \binom{n}{k} x^k (1-x)^{n-k}.$$

It will be shown that the operators M_n have the same saturation order and class as the well known Kantorovič operators P_n defined for a function $f \in L_1(I)$ by

$$(P_n f)(x) := (n+1) \sum_{k=0}^{n} p_{nk}(x) \int_{I_k} f(t)\, dt, n \in \mathbb{N}, \qquad I_k := \left[\frac{k}{n+1}, \frac{k+1}{n+1}\right].$$

The saturation properties of the P_n were investigated by Maier and Riemenschneider [6, 7, 8, 10].

For the proof of the direct theorem we use—as in [7, 10]—especially the L_p-approximation error for certain logarithmic functions. We were able to get the needed properties by using known estimates for the L_p-approximation error of the Kantorovič operators, theorems of Voronovskaja type for L_1-integrable functions, and by application of Lebesgue's dominated convergence theorem.

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The main result of this paper will be that for $f \in L_p(I)$, $1 \leqslant p < \infty$,

$$\|M_n f - f\|_p = O\left(\frac{1}{n+1}\right) \Leftrightarrow f \in S_p,$$

where for $f \in L_p(I)$, $1 \leqslant p \leqslant \infty$, S_p is defined by

$$S_p := \left\{ f \mid f(x) = k + \int_y^x \frac{h(t)}{t(1-t)}\, dt \text{ a.e. on } I,\ y \in (0, 1),\ h(0) = h(1) = 0, \right.$$

$$\left. h' \in L_p(I) \text{ for } 1 < p \leqslant \infty,\ h \in BV(I) \text{ for } p = 1 \right\}.$$

2. L_p-Approximation of Certain Logarithmic Functions

In this section we will obtain estimates for the L_p-approximation error of log-functions. First we need the following Voronovskaja type result for L_1-integrable functions proved in [3].

Lemma 1.1. *For $f \in L_1(I)$, f twice differentiable in a point $x \in (0, 1)$, there holds*

$$\lim_{n \to \infty} (n+1)(M_n f - f)(x) = (x(1-x) f'(x))'.$$

We now prove an analogous result for the Kantorovič operators P_n.

Lemma 1.2. *For $f \in L_1(I)$, f twice differentiable in a point $x \in (0, 1)$, there holds*

$$\lim_{n \to \infty} (n+1)(P_n f - f)(x) = \tfrac{1}{2}(x(1-x) f'(x))'.$$

Proof. Let $F(z) := \int_0^z f(t)\, dt$, then

$$F'(z) = f(z) \qquad \text{a.e. in } I$$

and

$$F^{(k)}(x) = f^{(k-1)}(x) \qquad \text{for } k = 1, 2, 3 \text{ as } f \text{ is twice differentiable in } x.$$

Consider now the Taylor formula

$$F(t) = F(x) + (t-x) F'(x) + \frac{1}{2}(t-x)^2 F''(x)$$

$$+ \frac{1}{3!}(t-x)^3 F'''(x) + (t-x)^3 r(t-x), \tag{1}$$

where

$$|r(t-x)| \leqslant M \text{ for } t \in I \qquad \text{and} \qquad \lim_{t \to x} r(t-x) = 0.$$

Differentiating this formula with respect to t we get a.e. in I

$$f(t) = f(x) + (t-x) f'(x) + \frac{1}{2}(t-x)^2 f''(x) + \frac{d}{dt}((t-x)^3 r(t-x)).$$

Multiplying this by $(n+1)$ and taking the operator P_n on both sides we get (see [9])

$$\begin{aligned}(n+1)(P_n f - f)(x) &= \frac{1}{2}(1-2x) f'(x) \\ &\quad + \frac{1}{2} f''(x) \left[x(1-x) \frac{n}{n+1} - \left(x(1-x) - \frac{1}{3} \right) \frac{1}{n+1} \right] \\ &\quad + (n+1) \left[P_n \left(\frac{d}{dt}((t-x)^3 r(t-x)) \right) \right](x).\end{aligned}$$

In the following the last term in the sum on the right side will be denoted by $(n+1)E^*(n, x)$. Hence the proposition will be proved if we show that

$$\lim_{n \to \infty} (n+1) E^*(n, x) = 0. \tag{2}$$

Using the well known relation between the Kantorovič operator P_n and the Bernstein operator B_{n+1} (see [5, p. 30]),

$$\frac{d}{du}(B_{n+1} F)(u) = (P_n f)(u),$$

and considering the result at the fixed point x, we get

$$\begin{aligned}(n+1) E^*(n, x) &= (n+1) \sum_{k=0}^{n+1} \left(\frac{k}{n+1} - x \right)^3 r\left(\frac{k}{n+1} - x \right) \\ &\quad \times \binom{n+1}{k} x^{k-1} (1-x)^{n-k} (k - (n+1)x).\end{aligned}$$

Consider now

$$\begin{aligned}&(n+1)\, x(1-x)\, E^*(n, x) \\ &\quad = \frac{1}{(n+1)^2} \sum_{k=0}^{n+1} p_{n+1,k}(x)(k-(n+1)x)^4 r\left(\frac{k}{n+1} - x \right).\end{aligned}$$

Choosing $\varepsilon>0$, there exists a $\delta>0$ so that $|r(t-x)|<\varepsilon$ for all $|t-x|<\delta$ and we obtain

$$\begin{aligned}
&(n+1)\,x(1-x)\,|E^*(n,x)| \\
&\leqslant \frac{1}{(n+1)^2}\left\{\sum_{|k/(n+1)-x|<\delta}(k-(n+1)x)^4 p_{n+1,k}(x)\left|r\left(\frac{k}{n+1}-x\right)\right|\right. \\
&\quad\left.+\frac{1}{(n+1)^2}\frac{1}{\delta^2}\sum_{|k/(n+1)-x|\geqslant\delta}(k-(n+1)x)^6 p_{n+1,k}(x)\left|r\left(\frac{k}{n+1}-x\right)\right|\right\} \\
&\leqslant \frac{1}{(n+1)^2}\left\{\varepsilon\sum_{k=0}^{n+1}(k-(n+1)x)^4 p_{n+1,k}(x)\right. \\
&\quad\left.+\frac{1}{(n+1)^2}\frac{M}{\delta^2}\sum_{k=0}^{n+1}(k-(n+1)x)^6 p_{n+1,k}(x)\right\} \\
&\leqslant x(1-x)\,K\left\{\varepsilon+\frac{1}{n+1}\frac{M}{\delta^2}\right\}
\end{aligned}$$

with a constant K independent of n and x. The last estimate follows by Lorentz's formulas (see [5, p. 14]).

From this we have (2) and the lemma is proved.

Note that we get from the above estimate

$$(n+1)\,|E^*(n,x)|\leqslant K, \tag{3}$$

where K is independent of n and x which will be used later.

For the analogous term referring to the operator M_n we get the following result:

LEMMA 1.3. *Let $f\in L_1(I)$, f twice differentiable in a point $x\in(0,1)$. Then the following estimate holds.*

$$(n+1)\,|E(n,x)|\leqslant K[1+(x(1-x))^{-1/2}]$$

with a constant K independent of n and x and

$$E(n,x)=\left[M_n\left(\frac{d}{dt}((t-x)^3 r(t-x))\right)\right](x),$$

where $(t-x)^3 r(t-x)$ is the same remainder term as in (1).

Proof. The proposition follows as a direct consequence of the estimates in [3]. Derriennic got by means of integration by parts

$$E(n, x) = (n+1)(x^n(1-x)^3 r(1-x) + (1-x)^n x^3 r(-x))$$
$$-(n+1) \sum_{k=0}^{n} p_{nk}(x) \int_I p'_{nk}(t)(t-x)^3 r(t-x)\, dt.$$

Obviously the first term on the right side equals $O(1/(n+1))$. The second term will be denoted by $e(n, x)$. There holds (see [3])

$$(x(1-x)\, ne(n, x))^2$$
$$\leqslant n^3 \left[\sum_{k=0}^{n-1} p_{n+1,k+1}(x)((n+1)\,x-(k+1))^2 \right] \left[\varepsilon^2 S_{n6}(x) + \frac{M}{\delta^2} S_{n8}(x) \right], \tag{4}$$

choosing ε and δ as in the proof of Lemma 1.2 and putting

$$S_{nm}(x) := \sum_{k=0}^{n-1} p_{n+1,k+1}(x) \int_I p_{n-1,k}(t)(t-x)^m\, dt. \tag{5}$$

Reference [3] gives us the estimate

$$S_{n,2m}(x) = O\left(\frac{1}{n^{m+1}}\right)$$

and by Lorentz's formulas (see [5, p. 14]) we have that the term in the first bracket on the right side of (4) is bounded by $(n+1)\,x(1-x)$. By summarizing these results we get the proposition of Lemma 1.3.

We are now able to estimate the L_p-approximation error for certain log-functions. From now on we define

$$g_1(t) := \ln(t), \qquad t \in (0, 1],$$
$$g_2(t) := \ln(1-t), \qquad t \in [0, 1),$$
$$g(t) := g_1(t) - g_2(t), \qquad t \in (0, 1).$$

LEMMA 1.4. *Let $1/q = (p-1)/p$ for $1 \leqslant p < \infty$ and $1/q = 1$ for $p = \infty$. Then the following statements are valid*:

(i) $\|x^{1/q}(M_n g_1 - g_1)\|_p = O\left(\frac{1}{n+1}\right)$,

(ii) $\|(1-x)^{1/q}(M_n g_2 - g_2)\|_p = O\left(\frac{1}{n+1}\right)$,

(iii) $\|x^{1/q}(1-x)^{1/q}(M_n g - g)\|_p = O\left(\frac{1}{n+1}\right)$.

Proof. First we remark that

$$\|x^{1/q}(M_n g_1 - g_1)\|_p = \|(1-x)^{1/q}(M_n g_2 - g_2)\|_p$$

holds by a change of variables and transformation of the summation index. Furthermore (iii) is a direct consequence of (i) and (ii). So we only have to prove (i). We will deal explicitly with $p=1$ and $p=\infty$ using interpolation methods (see also [10]) for the remaining cases.

$p=1$. As $g_1 \in L_1(I)$ and g_1 is twice differentiable in every point $x \in (0, 1]$, we get from the result in [3] by putting in the terms of g_1' and g_1''

$$(n+1)(M_n g_1 - g_1)(x) = \frac{(n+1)}{(n+2)(n+3)}\left[-\frac{1}{x^2}+\frac{6}{x}-(n+9)\right] + (n+1)\,E(n, x).$$

Thus we have

$$\begin{aligned} M_n^*(x) &:= (n+1)\left[(M_n g_1 - g_1)(x) - \frac{6(n+1)^2}{(n+2)(n+3)}(P_n g_1 - g_1)(x)\right] \\ &= \frac{2n(n+1)}{(n+2)(n+3)} + (n+1)\left[E(n, x) - \frac{6(n+1)^2}{(n+2)(n+3)}E^*(n, x)\right]. \end{aligned}$$

By (3) and Lemma 1.3 we get

$$|M_n^*(x)| \leqslant K(1+(x(1-x))^{-1/2}) =: s(x) \qquad \text{a.e. on } I. \tag{6}$$

This means that there exists a function $s \in L_1(I)$ which bounds $|M_n^*(x)|$ independent of n a.e. on I.

We are now able to change integration and taking the limit by use of Lebesgue's dominated convergence theorem,

$$\begin{aligned} &\lim_{n\to\infty} \|M_n^*\|_1 \\ &= \int_I \lim_{n\to\infty}\left|(n+1)(M_n g_1 - g_1)(x) - \frac{6(n+1)^2}{(n+2)(n+3)}(P_n g_1 - g_1)(x)\right| dx \\ &= \int_I \left|(1-2x)\frac{1}{x} - x(1-x)\frac{1}{x^2} - 3(1-2x)\frac{1}{x} + 3x(1-x)\frac{1}{x^2}\right| dx = 2. \end{aligned}$$

Hence we get from the definition of M_n^* and the fact that $\|P_n g_1 - g_1\|_1 = O(1/(n+1))$ (see [10]) the proposition for the case $p=1$

$$\|M_n g_1 - g_1\|_1 \leqslant \|M_n g_1 - g_1 - \frac{6(n+1)^2}{(n+2)(n+3)} (P_n g_1 - g_1)\|_1 + \frac{6(n+1)^2}{(n+2)(n+3)} \|P_n g_1 - g_1\|_1 = O\left(\frac{1}{n+1}\right).$$

Now we treat the case
$p = \infty$. We look at

$$\|x(M_n g_1 - g_1)\|_\infty \leqslant \|x(M_n g_1 - g_1)\|_\infty^{[0,1/2]} + \|x(M_n g_1 - g_1)\|_\infty^{[1/2,1]}.$$

Using the fact that $\|x(P_n g_1 - g_1)\|_\infty = O(1/(n+1))$ (see [10]) and the estimate (6) we obtain

$$\|x(M_n g_1 - g_1)\|_\infty^{[0,1/2]} = O\left(\frac{1}{n+1}\right). \tag{7}$$

Now we have to look at $x \in [\frac{1}{2}, 1]$.

By the same arguments as in the proof of Lemma 1.2 with the Taylor formula for $G(t) := \int_0^t g_1(z)\,dz$ we have

$$G(t) = G(x) + (t-x)\,G'(x) + \frac{1}{2}(t-x)^2\,G''(x) + \frac{1}{3!}(t-x)^3\,G'''(x) + \frac{1}{4!}(t-x)^4 G''''(x) + (t-x)^4 r(t-x),$$

where

$$|r(t-x)| \leqslant M,\ t \in [0,1] \qquad \text{and} \qquad \lim_{t \to x} r(t-x) = 0.$$

Differentiating this with respect to t and taking the operator M_n on both sides we get by using the recursion formulas for

$$\sum_{k=0}^{n} p_{nk}(x) \int_I p_{nk}(t)(t-x)^s\,dt, \qquad s \in \mathbb{N}_0$$

(see [2])

$$(n+1)\,|(M_n g_1 - g_1)(x)| \leqslant K + (n+1)\,|R(n,x)|$$

with

$$R(n,x) := \left[M_n\left(\frac{d}{dt}((t-x)^4\,r(t-x))\right)\right](x).$$

We have used the fact that $x \in [\frac{1}{2}, 1]$ implies the boundedness of $|g_1^{(k)}|$ for $k = 1, 2, 3$.

By the same arguments as in [3] we get

$$|R(n,x)| \leqslant (n+1)\,|x^n(1-x)^4\,r(1-x)+(1-x)^n\,x^4 r(-x)|$$
$$+(n+1)\left|\sum_{k=0}^{n} p_{nk}(x)\int_I p'_{nk}(t)(t-x)^4 r(t-x)\,dt\right|$$
$$=: r_1(x)+r_2(x)$$

and r_1 is bounded by $K/(n+1)$, K independent of n and x.

In an analogous way to [3] we obtain

$$(x(1-x)\,nr_2(x))^2 \leqslant n^3(n+1)\,x(1-x)\left[\varepsilon^2 S_{n8}(x)+\frac{M}{\delta^2}\,S_{n10}(x)\right], \qquad (8)$$

$S_{nm}(x)$ defined as in (5).

Using the notation

$$S^*_{nm}(x)=\sum_{k=0}^{n-1}\frac{1}{(k+1)(n-k)}\,p_{n-1,k}(x)\int_I p_{n-1,k}(t)(t-x)^m\,dt$$

we have

$$S_{nm}(x)=x(1-x)\,n(n+1)\,S^*_{nm}(x)$$

and we can deduce by straightforward calculation the recursion formula

$$S^*_{nm}(x)=\frac{1}{n+m}\,[x(1-x)(S^{*\prime}_{n,m-1}(x)$$
$$+2(m-1)\,S^*_{n,m-2}(x))+m(1-2x)\,S^*_{n,m-1}(x)]$$

and by estimating $S^*_{n0}(x)$ and $S^*_{n1}(x)$ we get

$$|S_{n8}(x)| \leqslant K_1\,\frac{x(1-x)}{n^4}, \qquad |S_{n10}(x)| \leqslant K_2\,\frac{x(1-x)}{n^5},$$

the constants K_1 and K_2 being independent of n and x.

From this we get with (8)

$$(x(1-x)\,nr_2(x))^2 \leqslant (x(1-x))^2\left[K_1\varepsilon^2+K_2\,\frac{M}{\delta^2 n}\right].$$

Hence $|r_2| = O(1/(n+1))$.

Thus it follows that

$$\|x(M_n g_1 - g_1)\|_\infty^{[1/2,1]} = O\left(\frac{1}{n+1}\right)$$

and taking account of (7) we have proved the case $p=\infty$. The cases $1<p<\infty$ now follow by using interpolation theory.

Now we possess the necessary tools for the proof of the main results.

3. Saturation of M_n

We first mention a direct theorem. We have proved the same estimates for log-functions concerning M_n as they were given for P_n in [7, 10].

Using the representation

$$(M_n f)(x) = \int_I K(n, t, x)\, f(t)\, dt,$$

where

$$K(n, t, x) := (n+1) \sum_{k=0}^{n} p_{nk}(x)\, p_{nk}(t)$$

we are now able to show the direct result in the same way as in [10].

Theorem 2.1. *Let $f \in L_p(I)$, $f \in S_p$. Then the following result holds:*

$$(n+1)\, \|M_n f - f\|_p \leqslant \begin{cases} C[\|f'\|_p + \|h'\|_p], & 1<p\leqslant\infty, \\ C[\|h\|_\infty + \|h\|_{BV}], & p=1, \end{cases}$$

C denoting a constant independent of n and x.

Proof. Using the representation

$$f(t) - f(x) = x(1-x)\, f'(x)(g(t) - g(x)) + \int_t^x (g(u) - g(t))\, dh(u)$$

we get by applying the operator M_n, then taking L_p-norms on both sides and using Lemma 2 in [10] ($f \in L_p(I)$, $f \in S_p$, $1<p\leqslant\infty$ implies $f' \in L_p(I)$)

$$\|M_n f - f\|_p \leqslant \left\| \left[M_n \left(\int_t^x (g(u) - g(t))\, dh(u) \right) \right] (x) \right\|_p + \begin{cases} \dfrac{C}{n+1} \|f'\|_p, & 1<p\leqslant\infty, \\ \dfrac{C}{n+1} \|h\|_\infty, & p=1, \end{cases}$$

where we write the variable x in the L_p-norm only for more clearness. Thus we now have to estimate the first term on the right side. Again we do this only for $p=1$ and $p=\infty$ using interpolation theory (see [10]) for the rest of the proposition.

$p=1$. By using Fubini's theorem and the fact that

$$\operatorname{sgn}(g(u)-g(t))=\operatorname{sgn}(u-t),$$

we get

$$\int_I \left|\left[M_n\left(\int_t^x (g(u)-g(t))\,dh(u)\right)\right](x)\right| dx$$

$$\leqslant \int_I \left\{\int_u^1 [M_n((g(u)-g(t))_+)](x)\,dx\right.$$

$$\left.+\int_0^u [M_n((g(t)-g(u))_+)](x)\,dx\right\} |dh(u)|.$$

We now show that the term in the curly bracket equals $O(1/(n+1))$ which completes the proof for the case $p=1$.

For every function $f\in L_1(I)$ it is easily seen that there holds

$$\int_I (M_n f-f)(x)\,dx=0.$$

As $(g(u)-g(x))_+=0$ for $u\leqslant x$, it follows that

$$0=\int_0^u [[M_n((g(u)-g(t))_+)](x)-(g(u)-g(x))_+]\,dx$$

$$+\int_u^1 [M_n((g(u)-g(t))_+)](x)\,dx.$$

Thus

$$\{...\}=\int_0^u (M_n g-g)(x)\,dx\leqslant\int_I |(M_n g-g)(x)|\,dx=O\left(\frac{1}{n+1}\right)$$

by Lemma 1.4(iii).

$p=\infty$. With analogous transformations as in [10] we get

$$\left|\left[M_n\left(\int_t^x (g(u)-g(t))\,h'(u)\,du\right)\right](x)\right|$$

$$\leqslant \|h'\|_\infty\{-x[(M_n g_1-g_1)(x)]-(1-x)[(M_n g_2-g_2)(x)]\}$$

$$=\|h'\|_\infty\, O\left(\frac{1}{n+1}\right)$$

by use of Lemma 1.4(i), (ii).

The proof of the following inverse theorem for the cases $1\leqslant p<\infty$ will be based on the investigation of special sequences of functionals.

THEOREM 2.2. *Let $f \in L_p(I)$, $1 \leqslant p < \infty$, and*

$$\|M_n f - f\|_p = O\left(\frac{1}{n+1}\right).$$

Then for the function f there holds $f \in S_p$.

Proof. We first look at the case

$p=1$. Let $f \in L_1(I)$ with $\|M_n f - f\|_1 = O(1/(n+1))$ and $(L_n)_{n \in \mathbb{N}}$ a sequence of linear functionals defined for all $\varphi \in C(I)$ by

$$L_n(\varphi) := \int_I (n+1)(M_n f - f)(x)\, \varphi(x)\, dx. \tag{9}$$

We now prove two propositions about their convergence.

PROPOSITION 2.2.1. *For all $\varphi \in C^2(I)$ there holds*

$$\lim_{n \to \infty} L_n(\varphi) = \int_I f(x)(x(1-x)\, \varphi'(x))'\, dx =: L(\varphi). \tag{10}$$

Proof. By [2, Lemma III.3, Theorem II.5] we have for all $\alpha, \beta \in L_1(I)$ the symmetric relation

$$\int_I (M_n \alpha)(x)\, \beta(x)\, dx = \int_I \alpha(x)(M_n \beta)(x)\, dx$$

and for all $\varphi \in C^2(I)$,

$$\lim_{n \to \infty} (n+1)(M_n \varphi - \varphi)(x) = (x(1-x)\, \varphi'(x))'$$

uniformly on I.

Using these facts we get

$$\begin{aligned}
&\lim_{n \to \infty} (n+1) \int_I (M_n f - f)(x)\, \varphi(x)\, dx \\
&= \lim_{n \to \infty} (n+1) \int_I (M_n \varphi - \varphi)(x) f(x)\, dx \\
&= \int_I f(x)[\lim_{n \to \infty} (n+1)(M_n \varphi - \varphi)(x)]\, dx \\
&= \int_I f(x)(x(1-x)\, \varphi'(x))'\, dx
\end{aligned}$$

and Proposition 2.2.1 is proved.

PROPOSITION 2.2.2. *For all $\varphi \in C(I)$ there exists a subsequence $(L_{n_k})_{k\in\mathbb{N}}$ of $(L_n)_{n\in\mathbb{N}}$ such that*

$$\lim_{k\to\infty} L_{n_k}(\varphi) = \int_I \varphi(x)\, dh(x) =: L^*(\varphi), \tag{11}$$

where $h \in BV(I)$ and $h(0) = h(1) = 0$.

Proof. As $\|M_n f - f\|_1 = O(1/(n+1))$, we get

$$|L_n(\varphi)| \leqslant \|\varphi\|_\infty (n+1)\, \|M_n f - f\|_1 \leqslant K \|\varphi\|_\infty$$

with a constant K independent of n. Thus the sequence $(L_n)_{n\in\mathbb{N}}$ is uniformly bounded.

By Riesz's representation theorem there exists for every L_n a unique $h_n \in BVN(I)$ such that

$$L_n(\varphi) = \int_I \varphi(x)\, dh_n(x) \text{ for all } \varphi \in C(I) \qquad \text{and} \qquad \|L_n\| = \bigvee_0^1 (h_n),$$

where $h \in BVN([a, b])$ if h is of bounded variation on $[a, b]$ and $h(a) = 0$. We show that

$$h_n = (n+1)(\bar{M}_n f - F),$$

where

$$F(x) = \int_0^x f(t)\, dt$$

and

$$(\bar{M}_n f)(x) := (n+1) \sum_{k=0}^n \int_0^x p_{nk}(t)\, dt \int_I p_{nk}(t) f(t)\, dt.$$

We have

$$\frac{d}{dx}((n+1)(\bar{M}_n f - F))(x) = (n+1)(M_n f - f)(x)$$

and $(n+1)(\bar{M}_n f - F) \in BVN(I)$ as $\bar{M}_n f$ is a polynomial of degree n, F is an absolutely continuous function on I, and $(\bar{M}_n f)(0) = F(0) = 0$. Therefore $h_n = (n+1)(\bar{M}_n f - F)$ as $h_n \in BVN(I)$ is unique.

From

$$\int_I p_{nk}(t)\, dt = \frac{1}{n+1}$$

we get $h_n(1)=0$. We have $\|L_n\| = \bigvee_0^1(h_n) = \|h_n\|_{BV}$ uniformly bounded and $|h_n(x)| = |h_n(x) - h_n(0)| \leqslant \|h_n\|_{BV}$.

Hence by two theorems of Helly and Bray (see [11, Theorem 16.3, Theorem 16.4]), there exists a subsequence $(L_{n_k})_{k \in \mathbb{N}}$ so that

$$\lim_{k \to \infty} L_{n_k}(\varphi) = \int_I \varphi(x)\, dh(x), \qquad h \in BV(I),\ h(0) = h(1) = 0$$

and Proposition 2.2.2 is proved.

From (10) and (11) we now have for all $\varphi \in C^2(I)$

$$\int_I f(x)(x(1-x)\,\varphi'(x))'\, dx = \int_I \varphi(x)\, dh(x), \tag{12}$$

where $h \in BV(I)$, $h(0) = h(1) = 0$.

This is the same equation as in Maier's proof for the L_1-saturation of the Kantorovič operators (see [7, (14)]). Hence

$$f(x) = k + \int_y^x \frac{h(t)}{t(1-t)}\, dt \qquad \text{a.e. on } I,\ k \in \mathbb{R},\ y \in (0, 1)$$

and the case $p = 1$ is proved.

We now look at

$1 < p < \infty$. Let $f \in L_p(I)$ with $\|M_n f - f\|_p = O(1/(n+1))$ and for $\varphi \in L_q(I)$ consider the sequence of functionals defined in (9).

The equality (10) still holds true and we get by Hölder's inequality

$$|L_n(\varphi)| \leqslant (n+1)\, \|M_n f - f\|_p\, \|\varphi\|_q \leqslant K \|\varphi\|_q \qquad \text{for all} \quad \varphi \in L_q(I)$$

as $\|M_n f - f\|_p = O(1/(n+1))$. This implies the uniformly boundedness of the sequence $(L_n)_{n \in \mathbb{N}}$.

As every ball of $L_q^*(I)$ is weakly*-compact, there exists a subsequence $(L_{n_k})_{k \in \mathbb{N}}$ which is weakly*-convergent to a functional L^* in $L_q(I)$. The representation theorem for bounded linear functionals in $L_q(I)$ gives us the existence of a function $h \in L_p(I)$ such that

$$L^*(\varphi) = \int_I h(x)\, \varphi(x)\, dx. \tag{13}$$

Now (10) equals (13) for all $\varphi \in C^2(I)$ and we have

$$\int_I f(x)(x(1-x)\,\varphi'(x))'\, dx = \int_I \varphi(x)\, h(x)\, dx. \tag{14}$$

The same equation was obtained by Maier in his proof for the L_p-saturation of the Kantorovič operators [8, (8)]. Hence $f \in S_p$ and the theorem is proved.

Theorems 2.1 and 2.2 now give a global saturation result for the operators M_n and we see that they have the same saturation order and class as the Kantorovič operators.

The trivial class follows as a direct consequence of the above proofs as the solutions of the homogeneous parts of the integral Eqs. (12) and (14).

COROLLARY 2.3. *For $f \in L_p(I)$, $1 \leqslant p < \infty$, there holds*

$$\|M_n f - f\|_p = o\left(\frac{1}{n+1}\right) \Leftrightarrow f = k \qquad \text{a.e. on } I \text{ where } k \in \mathbb{R}.$$

REFERENCES

1. M. M. DERRIENNIC, "Sur l'approximation des fonctions d'une ou plusieurs variables par des polynômes de Bernstein modifiés et application au problème des moments," Thèse de 3e cycle, Université de Rennes, 1978.
2. M. M. DERRIENNIC, Sur l'approximation de fonctions intégrables sur [0, 1] par des polynômes de Bernstein modifiés, *J. Approx. Theory* **31** (1981), 325–343.
3. M. M. DERRIENNIC, Additif au papier "Sur l'approximation de fonctions intégrables sur [0, 1] par des polynômes de Bernstein modifiés," Juin 1985 (unpublished).
4. J. L. DURRMEYER, "Une formule d'inversion de la transformée de Laplace-applications à la théorie des moments," Thèse de 3e cycle, Faculté des Sciences de l'Université de Paris, 1967.
5. G. G. LORENTZ, "Bernstein Polynomials," Univ. of Toronto Press, Toronto, 1953.
6. V. MAIER, "Güte- und Saturationsaussagen für die L_1-Approximation durch spezielle Folgen linearer positiver Operatoren," Dissertation, Universität Dortmund, 1976.
7. V. MAIER, The L_1 saturation class of the Kantorovič operator, *J. Approx. Theory* **22** (1978), 223–232.
8. V. MAIER, L_p-approximation by Kantorovič operators, *Anal. Math.* **4** (1978), 289–295.
9. A. MARLEWSKI, Asymptotic form of Bernstein-Kantorovič approximation, *Fasc. Math.* **12** (1980), 99–102.
10. S. D. RIEMENSCHNEIDER, The L_p-saturation of the Bernstein–Kantorovitch polynomials, *J. Approx. Theory* **23** (1978), 158–162.
11. D. V. WIDDER, "The Laplace Transform," Princeton Univ. Press, Princeton, NJ, 1946.